

# Cambridge International AS & A Level

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#### **FURTHER MATHEMATICS**

9231/21

Paper 2 Further Pure Mathematics 2

October/November 2021

2 hours

You must answer on the question paper.

You will need: List of formulae (MF19)

#### **INSTRUCTIONS**

- Answer all questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do not use an erasable pen or correction fluid.
- Do not write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

### **INFORMATION**

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has 16 pages.

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2 The matrix A is given by

$$\mathbf{A} = \begin{pmatrix} -1 & 2 & 12 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Use the characteristic equation of A to show that

$$\mathbf{A}^4 = p\mathbf{A}^2 + q\mathbf{I},$$

| where $p$ and $q$ are integers to be determined. | [6] |
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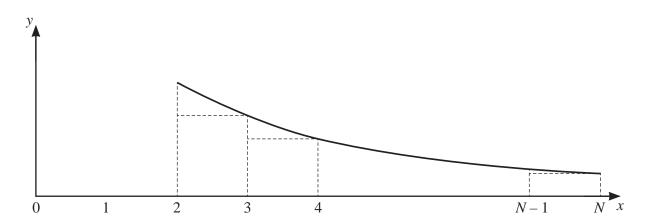
| 3 | The curve     | C has  | equation |
|---|---------------|--------|----------|
| • | I IIC CUI I C | CIIIUS | cquation |

$$xy^3 - 4x^3y = 3.$$

| Show that, at the point $(-1,1)$ on $C$ , $\frac{dy}{dx} = 11$ . |  |
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| <b>(b)</b> | Find the value of $\frac{d^2y}{dx^2}$ at the point $(-1,1)$ . | [5] |
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The diagram shows the curve with equation  $y = \frac{\ln x}{x^2}$  for  $x \ge 2$ , together with a set of (N-2) rectangles of unit width.

(a) By considering the sum of the areas of these rectangles, show that

| $\sum_{r=1}^{N} \frac{\ln r}{r^2} < \frac{2+3\ln 2}{4} - \frac{1+\ln N}{N}.$ | [7] |
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| b) | Use a similar method to find, in terms of N, a lower bound for $\sum_{r=1}^{N}$ | $\sum_{r=1}^{N} \frac{\ln r}{r^2}.$ [3] |
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5 Find the particular solution of the differential equation

| <u>.</u>  | $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 4\cos x,$ |
|---|--|
| given that, when $x = 0$ , $y = -4$ and $\frac{6}{6}$ | $\frac{\mathrm{d}y}{\mathrm{d}x} = 3. \tag{11}$  |
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6 (a) Use de Moivre's theorem to show that

| 50         | $\csc^5 \theta$  |       |
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| cosec 50 = | $= \frac{\csc^5 \theta}{5 \csc^4 \theta - 20 \csc^2 \theta + 16}.$ | . [6] |
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| ( | h | ) | Hence | ohtain | the | roots | $\alpha f$ | the | ea | mati | ion  |
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| ١ | v | , | Hence | Obtain | uic | 10018 | ΟI         | uic | cq | uai  | 1011 |

| 5             | 104  | $+40x^{2}$ | 22  | _ ^  |
|---------------|------|------------|-----|------|
| $x^{\circ}$ — | (0x) | $+40x^{2}$ | -32 | = () |

| in the form $\csc(q\pi)$ , where $q$ is rational. | [4] |
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| 7 | (a) | Show that an appropriate integrating factor for            |
|   |     | $\sqrt{x^2 - 1} \frac{dy}{dx} + y = x^2 - x\sqrt{x^2 - 1}$ |
|   |     | is $x + \sqrt{x^2 - 1}$ . [4]                              |
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| Hence find the solution of the differential equation                                 |  |  |  |
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| $\sqrt{x^2 - 1} \frac{dy}{dx} + y = x^2 - x\sqrt{x^2 - 1}$                           |  |  |  |
| for which $y = 1$ when $x = \frac{5}{4}$ . Give your answer in the form $y = f(x)$ . |  |  |  |
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|          | $2\cosh^2 A = \cosh 2A + 1.$   | [3]                 |
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|          | e $C$ has parametric equations   |                     |
|          | $x = 2\cosh 2t + 3t$ , $y = \frac{3}{2}\cosh 2t - 4t$ , for $-\frac{1}{2} \le t \le \frac{1}{2}$ . |                     |
| The area | of the surface generated when $C$ is rotated through $2\pi$ radians about the                      | e v-axis is denoted |
| by $A$ . |  | y uxis is denoted   |
| (b) (i)  | Show that $A = 10\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} (2\cosh 2t + 3t)\cosh 2t  dt$ .             | [4]                 |
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| Hence find $A$ in terms of $\pi$ and $e$ . | [7] |
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## **Additional Page**

| If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown. |  |  |  |  |
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